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von Kármán–Howarth equations for Hall magnetohydrodynamic flows

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The von Kármán–Howarth equations are derived for three-dimensional Hall magnetohydrodynamics in the case of a homogeneous and isotropic turbulence. From these equations, we derive exact scaling laws for the third-order correlation tensors. We show how these relations are compatible with previous heuristic and numerical results. These multiscale laws provide a relevant tool to investigate the nonlinear nature of the high-frequency magnetic field fluctuations in the solar wind or, more generally, in any plasma where the Hall effect is important.

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Turbulence remains one of the last great unsolved problems in classical physics which has evaded physical understanding and systematic description for many decades. For that reason, any exact results appear almost as a miracle. In his third 1941 turbulence paper, Kolmogorov found that an exact and nontrivial relation may be derived from Navier-Stokes equations—which can be seen as the archetype equations for describing turbulence—for the third-order longitudinal structure function [1]. Because of the rarity of such results, Kolmogorov's four-fifths law is considered as one of the most important results in turbulence [2].

The derivation of Kolmogorov's law uses earlier exact results found by von Kármán and Howarth in 1938 [3]: it is the well-known von Kármán–Howarth (vKH) equation that describes the dynamical evolution of the second-order correlation tensors. Very few extensions of such results (vKH equations and four-fifths law) to other fluids have been made; it concerns scalar passively advected [4], such as the temperature or a pollutant in the atmosphere, and astrophysical magnetized fluid described in the framework of magnetohydrodynamics (MHD) [5,6]. The addition in the analysis of the magnetic field and its coupling with the velocity field renders the problem more difficult and, in practice, we are dealing with a couple of equations.

Signatures of turbulence in astrophysical flows are found in the solar wind [7], the interstellar [8], galactic, and even intergalactic media [9]. In the case of the interplanetary medium, we have access to very precise *in situ* measurements which show, in particular, the existence of a steepening of the magnetic field fluctuation spectrum at frequencies higher than 1 Hz [10,11] whose origin may be attributed to nonlinear Hall-MHD processes [12]. Efforts from observers are currently made to show the presence of intermittency at high frequency. In this quest, any theoretical or model predictions about moderate or high-order correlation tensors is particularly important for the understanding of solar wind and, more generally speaking, astrophysical turbulence.

In this Rapid Communication, we derive the vKH equations for three-dimensional (3D) Hall-MHD fluids. From these exact results, we give the equivalent of Kolmogorov's four-fifths law for the velocity, magnetic, and current density field correlations.

We start our analysis with the 3D incompressible Hall MHD equations

$$(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P_* + \mathbf{b}\cdot\nabla\mathbf{b} + \nu\Delta\mathbf{v}, \tag{1}$$

$$(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{v} - d_I\nabla\times[(\nabla\times\mathbf{b})\times\mathbf{b}] + \eta\Delta\mathbf{b}, \tag{2}$$

with $\nabla\cdot\mathbf{v}=0$ and $\nabla\cdot\mathbf{b}=0$. The magnetic field $\mathbf{b}$ is normalized to a velocity ($\mathbf{b}\to\sqrt{\mu_0 n m_i}\mathbf{b}$, with m_i the ion mass and n the electron density), $\mathbf{v}$ is the plasma flow velocity, P_* is the total (magnetic plus kinetic) pressure, ν is the viscosity, η is the magnetic diffusivity, and d_I is the ion inertial length ($d_I = c/\omega_{pi}$, where c is the speed of light and ω_{pi} is the ion plasma frequency). Equations (1) and (2) may be rewritten more compactly as

$$\partial_t v_i = -\partial_i P_* + b_\ell\partial_\ell b_i - v_\ell\partial_\ell v_i + \nu\partial^2_{\ell\ell}v_i, \tag{3}$$

$$\partial_t b_i = b_\ell\partial_\ell v_i - v_\ell\partial_\ell b_i + d_I(J_\ell\partial_\ell b_i - b_\ell\partial_\ell J_i) + \eta\partial^2_{\ell\ell}b_i, \tag{4}$$

where $\mathbf{J}=\nabla\times\mathbf{b}$ is the normalized current density. Note the use of Einstein's notation. We see immediately that the third-order tensors that will appear in our analysis will be a combination of the velocity, the magnetic field, and the current density. This makes an important difference with Navier-Stokes fluids for which the tensor used to derive the four-fifths law is built with the same (velocity) field. As shown below, it will have a direct impact on the kinematics.

Before deriving the vHK relations for Hall-MHD fluids, one needs to introduce the kinematics adapted to this problem. The second-order correlation tensors, in the full isotropic and homogeneous case, may be written as [13]

$$R^X_{ij}(\mathbf{r}) = \langle X_i(\mathbf{x})X_j(\mathbf{x}')\rangle = F^X r_i r_j + G^X\delta_{ij}, \tag{5}$$

where $\mathbf{x}'=\mathbf{x}+\mathbf{r}$ and $X=v$ or b. F^X and G^X are four arbitrary functions of r^2 which will be specified later. The divergence-free condition $\partial_{r_j}R^X_{ij}(\mathbf{r})=0$ on the velocity and the magnetic field leads to the relations

$$4F^X + r\partial_r F^X + r^{-1}\partial_r G^X = 0, \tag{6}$$

which will be used later. We now introduce the longitudinal and lateral functions as, respectively,

$$R^X_{\parallel\parallel} = X^2 f^X(r) \quad \text{and} \quad R^X_{\perp\perp} = X^2 g^X(r). \tag{7}$$

The reference direction is the vector separation $\mathbf{r}$ such that, for example, the parallel component is the one along $\mathbf{r}$. The correlation functions $f^{v,b}$ and $g^{v,b}$ are mainly decreasing (see [13] for more details about the function g) and satisfy the condition $f^{v,b}(0)=g^{v,b}(0)=1$. From relations (6) and (7), we obtain

$$R_{ij}^X(\mathbf{r}) = X^2\left[f^X\delta_{ij} + \left(\frac{r}{2}\delta_{ij} - \frac{r_ir_j}{2r}\right)\partial_r f^X\right]. \tag{8}$$

The third-order correlation tensors that will appear in our derivation are [13]

$$\begin{aligned} S_{ijk}^1(\mathbf{r}) &= \langle v_i(\mathbf{x})v_j(\mathbf{x})v_k(\mathbf{x}')\rangle \\ &= A_1r_ir_jr_k + B_1(r_i\delta_{jk}+r_j\delta_{ik}) + D_1r_k\delta_{ij}, \end{aligned} \tag{9}$$

$$\begin{aligned} S_{ijk}^2(\mathbf{r}) &= \langle b_i(\mathbf{x})b_j(\mathbf{x})v_k(\mathbf{x}')\rangle \\ &= A_2r_ir_jr_k + B_2(r_i\delta_{jk}+r_j\delta_{ik}) + D_2r_k\delta_{ij}, \end{aligned} \tag{10}$$

$$\begin{aligned} S_{ijk}^3(\mathbf{r}) &= \langle v_i(\mathbf{x})b_j(\mathbf{x})b_k(\mathbf{x}')\rangle \\ &= A_3r_ir_jr_k + B_3r_i\delta_{jk} + C_3r_j\delta_{ik} + D_3r_k\delta_{ij}, \end{aligned} \tag{11}$$

$$\begin{aligned} S_{ijk}^4(\mathbf{r}) &= \langle J_i(\mathbf{x})b_j(\mathbf{x})b_k(\mathbf{x}')\rangle \\ &= A_4r_ir_jr_k + B_4r_i\delta_{jk} + C_4r_j\delta_{ik} + D_4r_k\delta_{ij}, \end{aligned} \tag{12}$$

where A_m, B_m, C_m, and D_m are arbitrary functions of r^2. Note that the last two tensors are *not* symmetric in the suffixes i and j, which makes an important difference with Navier-Stokes fluids where only the velocity field is used to build the third-order correlation tensor. The direct consequence is that we need not three but four arbitrary functions to define these tensors initially. In the same way as before, we use the continuity condition $\partial_{r_k}S_{ijk}^m(\mathbf{r})=0$ to constrain our system [13]; it leads to the relations

$$r\partial_r A_{1,2} + 5A_{1,2} + 2r^{-1}\partial_r B_{1,2} = 0, \tag{13}$$

$$r\partial_r D_{1,2} + 3D_{1,2} + 2B_{1,2} = 0, \tag{14}$$

$$r\partial_r A_{3,4} + 5A_{3,4} + r^{-1}\partial_r B_{3,4} + r^{-1}\partial_r C_{3,4} = 0, \tag{15}$$

$$r\partial_r D_{3,4} + 3D_{3,4} + B_{3,4} + C_{3,4} = 0. \tag{16}$$

Additionally, we note that $S_{iik}^m(\mathbf{r})=0$ whatever the value of m is (since it is a solenoidal first-order isotropic tensor); it gives the relations

$$A_{1,2}r^2 + 2B_{1,2} + 3D_{1,2} = 0, \tag{17}$$

$$A_{3,4}r^2 + B_{3,4} + C_{3,4} + 3D_{3,4} = 0. \tag{18}$$

We introduce now the basic functions which include the parallel and perpendicular components of the fields. We have

$$S_{\parallel\parallel\parallel}^{1,2} = A_{1,2}r^3 + (2B_{1,2}+D_{1,2})r = Y_{1,2}K_{1,2}(r), \tag{19}$$

$$S_{\perp\perp\parallel}^{1,2} = D_{1,2}r = Y_{1,2}h_{1,2}(r), \tag{20}$$

$$S_{\parallel\perp\perp}^{1,2} = B_{1,2}r = Y_{1,2}q_{1,2}(r), \tag{21}$$

$$S_{\parallel\parallel\parallel}^{3,4} = A_{3,4}r^3 + (B_{3,4}+C_{3,4}+D_{3,4})r = Y_{3,4}K_{3,4}(r), \tag{22}$$

$$S_{\perp\perp\parallel}^{3,4} = D_{3,4}r = Y_{3,4}h_{3,4}(r), \tag{23}$$

$$S_{\parallel\perp\perp}^{3,4} = B_{3,4}r = Y_{3,4}q_{3,4}(r), \tag{24}$$

$$S_{\perp\parallel\perp}^{3,4} = C_{3,4}r = Y_{3,4}s_{3,4}(r), \tag{25}$$

where K_m, h_m, q_m, and s_m are odd scalar functions and $Y_1=v^3$, $Y_2=Y_3=vb^2$, and $Y_4=Jb^2$. Conditions (13)–(18) simplify the expression of the third-order tensors, which write finally as

$$\begin{aligned} S_{ijk}^{1,2}(\mathbf{r}) = Y_{1,2}\Bigg[&\left(\frac{K_{1,2}-r\partial_r K_{1,2}}{2r^3}\right)r_ir_jr_k + \left(\frac{2K_{1,2}+r\partial_r K_{1,2}}{4r}\right) \\ &\times(r_i\delta_{jk}+r_j\delta_{ik}) - \frac{K_{1,2}}{2r}r_k\delta_{ij}\Bigg], \end{aligned} \tag{26}$$

$$\begin{aligned} S_{ijk}^{3,4}(\mathbf{r}) = Y_{3,4}\Bigg[&\left(\frac{K_{3,4}-r\partial_r K_{3,4}}{2r^3}\right)r_ir_jr_k + \frac{q_{3,4}}{r}r_i\delta_{jk} \\ &+ \left(\frac{K_{3,4}+r\partial_r K_{3,4}/2 - q_{3,4}}{r}\right)r_j\delta_{ik} - \frac{K_{3,4}}{2r}r_k\delta_{ij}\Bigg]. \end{aligned} \tag{27}$$

The first main goal of this paper is the derivation of the vKH equations for 3D Hall MHD fluids. We start with Eqs. (3) and (4) and use previous relations derived from the kinematics to find

$$\begin{aligned} \partial_t R_{ij}^v(\mathbf{r}) &= \langle v_i\partial_t v_j'\rangle + \langle v_j'\partial_t v_i\rangle \\ &= \langle v_ib_\ell'\partial_\ell' b_j'\rangle - \langle v_iv_\ell'\partial_\ell' v_j'\rangle - \langle v_i\partial_j' P'\rangle \\ &\quad + \langle v_j'b_\ell\partial_\ell b_i\rangle - \langle v_j'v_\ell\partial_\ell v_i\rangle - \langle v_j'\partial_i P\rangle \\ &\quad + \nu\langle v_i\partial'^2_{\ell\ell}v_j'\rangle + \nu\langle v_j'\partial^2_{\ell\ell}v_i\rangle, \end{aligned} \tag{28}$$

$$\begin{aligned} \partial_t R_{ij}^b(\mathbf{r}) &= \langle b_i\partial_t b_j'\rangle + \langle b_j'\partial_t b_i\rangle \\ &= \langle b_ib_\ell'\partial_\ell' v_j'\rangle - \langle b_iv_\ell'\partial_\ell' b_j'\rangle + \langle b_j'b_\ell\partial_\ell v_i\rangle \\ &\quad - \langle b_j'v_\ell\partial_\ell b_i\rangle + d_I(\langle b_iJ_\ell'\partial_\ell' b_j'\rangle - \langle b_ib_\ell'\partial_\ell' J_j'\rangle \\ &\quad + \langle b_j'J_\ell\partial_\ell b_i\rangle - \langle b_j'b_\ell\partial_\ell J_i\rangle) + \eta\langle b_i\partial'^2_{\ell\ell}b_j'\rangle \\ &\quad + \eta\langle b_j'\partial^2_{\ell\ell}b_i\rangle. \end{aligned} \tag{29}$$

After simple manipulations where we use, in particular, the divergence-free condition and the homogeneity assumption, we get

$$\partial_t R_{ij}^v = \partial_{r_\ell}(S_{i\ell j}^1 + S_{j\ell i}^1 - S_{i\ell j}^2 - S_{j\ell i}^2) + 2\nu\partial^2_{r_\ell r_\ell}R_{ij}^v, \tag{30}$$

$$\begin{aligned} \partial_t R_{ij}^b = \partial_{r_\ell}(&S_{\ell ji}^3 - S_{j\ell i}^3 + S_{\ell ij}^3 - S_{i\ell j}^3 + d_I S_{j\ell i}^4 - d_I S_{\ell ji}^4 \\ &+ d_I S_{i\ell j}^4 - d_I S_{\ell ij}^4) + 2\eta\partial^2_{r_\ell r_\ell}R_{ij}^b. \end{aligned} \tag{31}$$

Note that the pressure terms are suppressed because of isotropy [13]. These general dynamical equations reduce to a simple form for the diagonal part of the energy tensor,

$$\partial_t R_{ii}^v = 2\partial_{r_\ell}(S_{i\ell i}^1 - S_{i\ell i}^2) + 2\nu\partial^2_{r_\ell r_\ell}R_{ii}^v, \tag{32}$$

$$\partial_t R_{ii}^b = 2\partial_{r_\ell}(S_{\ell ii}^3 - S_{i\ell i}^3 + d_I(S_{i\ell i}^4 - S_{\ell ii}^4)) + 2\eta\partial^2_{r_\ell r_\ell}R_{ii}^b. \tag{33}$$

It is the basic equations from which it will be possible to derive the equivalent of the vKH relations. The introduction of Eqs. (26) and (27) into Eqs. (32) and (33) gives

$$\partial_t R^v_{ii} = \partial_{r_\ell}\left(\frac{v^3}{r} r_\ell(4+r\partial_r)K_1 - \frac{vb^2}{r} r_\ell(4+r\partial_r)K_2\right) + 2\nu\partial^2_{r_\ell r_\ell} R^v_{ii}, \tag{34}$$

$$\partial_t R^b_{ii} = \partial_{r_\ell}\left(-\frac{4vb^2}{r} r_\ell(K_3 + r\partial_r K_3/2 - 2q_3) + \frac{4d_I J b^2}{r} r_\ell(K_4 + r\partial_r K_4/2 - 2q_4)\right) + 2\eta\partial^2_{r_\ell r_\ell} R^b_{ii}. \tag{35}$$

By introducing

$$\widetilde{K}_m = \frac{1}{r^4}\partial_r(r^4 K_m), \quad \widetilde{K}_n = \frac{K_n + r\partial_r K_n/2 - 2q_n}{r}, \tag{36}$$

for $m=(1,2)$ and $n=(3,4)$, we obtain

$$\partial_t((3+r\partial_r)f^v v^2) = v^3\partial_{r_\ell}(r_\ell \widetilde{K}_1) - vb^2\partial_{r_\ell}(r_\ell\widetilde{K}_2) + 2\nu\partial^2_{r_\ell r_\ell}((3+r\partial_r)f^v v^2), \tag{37}$$

$$\partial_t((3+r\partial_r)f^b b^2) = 4d_I J b^2\partial_{r_\ell}(r_\ell \widetilde{K}_4) - 4vb^2\partial_{r_\ell}(r_\ell\widetilde{K}_3) + 2\eta\partial^2_{r_\ell r_\ell}((3+r\partial_r)f^b b^2). \tag{38}$$

By noting the identity (for isotropic turbulence) $\partial^2_{r_\ell r_\ell} = \partial^2_{rr} + (2/r)\partial_r$ and the more subtle relation

$$\left(\partial^2_{rr} + \frac{2}{r}\partial_r\right)(3+r\partial_r) = (3+r\partial_r)\frac{1}{r^4}\partial_r(r^4\partial_r), \tag{39}$$

we finally obtain after some simple manipulations

$$\partial_t((3+r\partial_r)f^v v^2) = v^3(3+r\partial_r)\widetilde{K}_1 - vb^2(3+r\partial_r)\widetilde{K}_2 + 2\nu(3+r\partial_r)\frac{1}{r^4}\partial_r(r^4\partial_r f^v v^2), \tag{40}$$

$$\partial_t((3+r\partial_r)f^b b^2) = 4d_I J b^2(3+r\partial_r)\widetilde{K}_4 - 4vb^2(3+r\partial_r)\widetilde{K}_3 + 2\eta(3+r\partial_r)\frac{1}{r^4}\partial_r(r^4\partial_r f^b b^2). \tag{41}$$

A first integral of these equations is

$$\partial_t f^v v^2 = v^3\widetilde{K}_1 - vb^2\widetilde{K}_2 + \frac{2\nu}{r^4}\partial_r(r^4\partial_r f^v v^2), \tag{42}$$

$$\partial_t f^b b^2 = -4vb^2\widetilde{K}_3 + 4d_I J b^2\widetilde{K}_4 + \frac{2\eta}{r^4}\partial_r(r^4\partial_r f^b b^2). \tag{43}$$

These exact equations are the vKH relations for 3D Hall MHD. It is the first main result of this paper.

We shall derive now the equivalent of the four-fifths law found by Kolmogorov for Navier-Stokes fluids [1]. We note the relation $R^X_{\parallel\,\parallel}(\mathbf{r}) = \langle X^2_\parallel\rangle - \frac{1}{2}B^X_{\parallel\,\parallel}(\mathbf{r})$, with the general form for the structure function

$$B^X_{ij}(\mathbf{r}) = \langle[X_i(\mathbf{x}') - X_i(\mathbf{x})][X_j(\mathbf{x}') - X_j(\mathbf{x})]\rangle, \tag{44}$$

where $X=(v,b)$. Introducing this relation into the vKH equations (42) and (43), we get

$$\partial_t\langle v^2_\parallel\rangle - \frac{1}{2}\partial_t B^v_{\parallel\,\parallel} = v^3\widetilde{K}_1 - vb^2\widetilde{K}_2 + \frac{2\nu}{r^4}\partial_r\left(r^4\partial_r\left(\langle v^2_\parallel\rangle - \frac{1}{2}B^v_{\parallel\,\parallel}\right)\right), \tag{45}$$

$$\partial_t\langle b^2_\parallel\rangle - \tfrac{1}{2}\partial_t B^b_{\parallel\,\parallel} = 4d_I J b^2\widetilde{K}_4 - 4vb^2\widetilde{K}_3 + \frac{2\eta}{r^4}\partial_r\left(r^4\partial_r\left(\langle b^2_\parallel\rangle - \frac{1}{2}B^b_{\parallel\,\parallel}\right)\right), \tag{46}$$

and eventually

$$\partial_t\langle v^2_\parallel\rangle - \partial_t\frac{B^v_{\parallel\,\parallel}}{2} = v^3\widetilde{K}_1 - vb^2\widetilde{K}_2 - \frac{\nu}{r^4}\partial_r(r^4\partial_r B^v_{\parallel\,\parallel}), \tag{47}$$

$$\partial_t\langle b^2_\parallel\rangle - \partial_t\frac{B^b_{\parallel\,\parallel}}{2} = 4d_I J b^2\widetilde{K}_4 - 4vb^2\widetilde{K}_3 - \frac{\eta}{r^4}\partial_r(r^4\partial_r B^b_{\parallel\,\parallel}). \tag{48}$$

We define the mean (total) energy dissipation rate per unit mass, ε^T, for isotropic turbulence, as

$$\partial_t\langle v^2_\parallel + b^2_\parallel\rangle = -(2/3)\varepsilon^T. \tag{49}$$

Exact scaling laws for third-order correlation tensors may be derived from the previous relations (47) and (48) by assuming the following assumptions specific to fully developed turbulence [2]. We first consider the long-time limit for which a stationary state is reached with a finite ε^T. Second, we take the infinite (magnetic) Reynolds number limit ($\nu\to 0$ and $\eta\to 0$) for which the mean energy dissipation rate per unit mass tends to a finite positive limit. Therefore, in the inertial range, we obtain at first order the relation

$$-\tfrac{1}{6}\varepsilon^T r = v^3(K_1 + r\partial_r K_1/4) - vb^2(K_2 + r\partial_r K_2/4) - vb^2(K_3 + r\partial_r K_3/2 - 2q_3) + d_I J b^2(K_4 + r\partial_r K_4/2 - 2q_4), \tag{50}$$

which can also be written as

$$-\tfrac{1}{6}\varepsilon^T r = (S^1_{\parallel\perp\perp} + \tfrac{1}{2}S^1_{\parallel\,\parallel\,\parallel}) - (S^2_{\parallel\perp\perp} + \tfrac{1}{2}S^2_{\parallel\,\parallel\,\parallel}) + (S^3_{\parallel\perp\perp} - S^3_{\perp\parallel\perp}) - d_I(S^4_{\parallel\perp\perp} - S^4_{\perp\parallel\perp}). \tag{51}$$

The last step consists in introducing structure functions which gives, after some manipulations, the final result

$$-\tfrac{4}{3}\varepsilon^T r = B^{vvv}_{\parallel ii} + B^{vbb}_{\parallel ii} - 2B^{bvb}_{\parallel ii} - 4d_I(S^4_{\parallel ii} - S^4_{i\parallel i}) = B^{vvv}_{\parallel ii} + B^{vbb}_{\parallel ii} - 2B^{bvb}_{\parallel ii} + 4d_I\langle[(\mathbf{J}\times\mathbf{b})\times\mathbf{b}']_\parallel\rangle, \tag{52}$$

with $B^{\alpha\beta\gamma}_{ijk} = \langle(\alpha'_i - \alpha_i)(\beta'_j - \beta_j)(\gamma'_k - \gamma_k)\rangle$.

Equations (42), (43), and (52) are the main results of this paper. The former equations are exact for homogeneous and isotropic turbulence, and the latter assumed additionally the existence of a large inertial range on which the total energy flux is finite and constant. The most remarkable aspect of these laws is that they not only provide a linear scaling for the third-order correlation tensors within the inertial range of length scales, but they also fix the value of the numerical factor appearing in front of the scaling relations. Another important remark is about the fields used to build the third-order correlation tensors. Indeed, the convenient variables are not only the velocity and magnetic field components but also the current density components. Note that attempts to find a simple expression in terms of only structure functions failed, and therefore relations (52) seem to be the most appropriate. A similar situation was found, for example, in MHD flows when the magnetic helicity is included in the analysis [6].

The vKH equations (42) and (43) derived here in the framework of Hall MHD are compatible with the one derived by Chandrasekhar [5] for MHD when the large-scale limit ($d_I \to 0$) is taken. Note that some minor manipulations have to be made in [5] to prove the compatibility since the notations are not the same (for example, we have $P \equiv vb^2\tilde{K}_3$). As explained above, the notation used here seems to be more suitable for Hall MHD, which therefore requires for a better understanding a complete rederivation of the dynamical equations. In the same way, when the large-scale limit is taken, relation (52) is compatible with previous works [6], which are also compatible with Navier-Stokes fluids when additionally the magnetic field is taken equal to zero.

The exact results found here provide a better theoretical understanding of Hall-MHD flows. They show that the scaling relation does not change its power dependence in the separation r at small scales if the statistical correlation tensor used is modified. The interesting point to note is the compatibility with previous heuristic and numerical results [14]. Indeed, a simple dimensional analysis gives the relations $r \sim b^3$ for large scales and $r^2 \sim b^3$ for small scales (since $J \sim b/r$), which give, respectively, the magnetic energy spectra $E \sim k^{-5/3}$ and $E \sim k^{-7/3}$. Therefore and contrary to appearances, the exact results found may provide a double-scaling relation.

These multiscale laws provide a relevant tool to investigate the nonlinear nature of the high-frequency magnetic field fluctuations in the solar wind whose (dissipative versus dispersive) origin is still controversial [12,15]. The use of multipoint data may give information about both the magnetic field and the current density which can be used to check the theoretical scaling relations. The observation of such a scaling law would be an additional piece of evidence for the presence of a dispersive inertial range and therefore for the turbulent nature of the high-frequency magnetic field fluctuations. The recent observation of the Yaglom MHD scaling law [16] at low frequency provides direct evidence for the presence of an inertial energy cascade in the solar wind. The theoretical results given here allow us now to extend this type of analysis to high-frequency magnetic field fluctuations and, more generally speaking, to better understand the role of the Hall effect in astrophysics, like, e.g., for the magnetorotational instability in cool protostellar disks [17] or in laboratory fusion plasmas. In such situations, isotropy is often broken because of the presence of a strong large-scale magnetic field (see, e.g., [18]) and a generalization of the present description to anisotropic turbulence is then necessary.

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